Pendulums and Elliptic Integrals

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1. Introduction

Many years ago before the advent of the “PC on every desktop” age, I became fascinated with the design of LC\(^1\) elliptic filters. As part of that endeavor, I also became intimately acquainted with elliptic integrals. Having an equal intrigue for numerical precision, I found that computing the elliptic integrals with high accuracy was very difficult if simple integration methods like Simpson’s Rule or Gaussian quadrature were resorted to. Thus began my search for a precision method of computations.

Some readers will no doubt be familiar with the solution path involved, but to those who are not, I invite you to read on.

2. Where Hence Elliptic Integrals?

Elliptic integrals show up in many places, electronic elliptic filters for one. One of the situations where people encounter them first is in connection with simple pendulum motion.

![Figure 1 Classical Pendulum](image)

A classical pendulum is shown in Figure 1 where

\[ m \text{ mass of pendulum} \]
\[ R \text{ length of pendulum} \]
\[ g \text{ acceleration of gravity (e.g., } 9.81 \text{ m/s}^2 \] 
\[ \alpha \text{ starting angle} \]

If we assume that the pendulum arm itself is both rigid and of zero mass, it is convenient to think about the motion of the pendulum bob in terms of motion along the fixed radius \( R \) where the angle \( \phi \) is a function of time. The tangential force perpendicular to \( R \) that the weight of the bob creates is given by

\[ F_T = mg \sin(\phi) \quad (1) \]

From Newton’s Laws of motion, this tangential force must be associated with a tangential acceleration which can be written as

\[ F_T = ma_T = m \left( \frac{dv_T}{dt} \right) = m \frac{d}{dt} \left( R \frac{d\phi}{dt} \right) \]
\[ = mR \frac{d^2\phi}{dt^2} \quad (2) \]

Proper attention to signs for the forces involved results in the describing differential equation in terms of \( \phi \) given as

\[ \frac{d^2\phi}{dt^2} - \frac{g}{R} \sin(\phi) = 0 \quad (3) \]

If the angular extents allowed for the pendulum swing are kept small, we can approximate \( \sin(\phi) \approx \phi \) which leads to the very simple differential equation

\[ \frac{d^2\phi}{dt^2} - \frac{g}{R} \phi = 0 \quad (4) \]

If we now hypothesize that the solution to this differential equation is given by \( \phi(t) = A \sin(\omega_0 t) \) and substitute into (4), we quickly see that this is indeed the correct solution with

\[ \omega_0 = \sqrt{\frac{g}{R}} \quad (5) \]

Returning now to the original nonlinear differential equation (3), this can be pursued further by multiplying both sides of the equation by \( d\theta / dt \) which creates

\[^1\) LC for inductor-capacitor
\[
\left(\frac{d^2\varphi}{dt^2}\right) \frac{d\varphi}{dt} = \omega_0^2 \sin(\varphi) \frac{d\varphi}{dt} \tag{6}
\]

and integrating both sides with respect to time results in

\[
\frac{1}{2} \left( \frac{d\varphi}{dt} \right)^2 - \omega_0^2 \cos(\varphi) = k \tag{7}
\]

where \( k \) is a constant of integration. Assuming that the pendulum has a maximal displacement of angle \( \varphi = \alpha \), then \( \varphi'(\alpha) = 0 \), and solving for the derivative and taking the positive root leads to

\[
\frac{d\varphi}{dt} = \omega_0 \sqrt{2 \left[ \cos(\varphi) - \cos(\alpha) \right]} \tag{8}
\]

Integrating one more time produces

\[
\int \frac{d\varphi}{\sqrt{2 \left[ \cos(\varphi) - \cos(\alpha) \right]}} = \omega_0 t \tag{9}
\]

The time required for \( \varphi \) to increase from 0 to \( \alpha \) is

\[
T = \frac{\sqrt{R}}{4 \sqrt{2g}} \int_{0}^{\alpha} \frac{d\varphi}{\sqrt{\cos(\varphi) - \cos(\alpha)}} \tag{10}
\]

Using the identities \( \cos(\varphi) = 1 - 2\sin^2(\varphi/2) \) and \( \cos(\alpha) = 1 - 2\sin^2(\alpha/2) \) in (10) leads to

\[
T = 2 \sqrt{\frac{R}{g}} \int_{0}^{\alpha} \frac{d\varphi}{\sqrt{k^2 - \sin^2(\varphi/2)}} \tag{11}
\]

with \( k = \sin(\alpha/2) \). A new variable can be defined as \( \sin(\varphi/2) = k \sin(\theta) \) from which

\[
\cos\left(\frac{\varphi}{2}\right) \frac{d\varphi}{2} = k \cos(\theta) d\theta \tag{12}
\]

which upon re-arrangement gives

\[
\frac{d\varphi}{\cos\left(\frac{\varphi}{2}\right)} = \frac{2k \cos(\theta) d\theta}{\cos\left(\frac{\varphi}{2}\right)} = \frac{2\sqrt{k^2 - \sin^2(\varphi/2)}}{\sqrt{1 - k^2 \sin^2(\theta)}} \tag{13}
\]

Substituting (13) into (11) leads finally to

\[
T = 4 \sqrt{\frac{R}{g}} \int_{0}^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2(\theta)}} \tag{14}
\]

The integral involved in (14) is an elliptic integral of the first kind. With \( k = \sin(\alpha/2) \), the integral is very well behaved because \( k \) is always < \( \sqrt{2}/2 \). In the case of elliptic filter usage however, \( k \) is often very close to unity thereby making numerical evaluation of (14) considerably more challenging.

**Aside:** Conservation of energy may be used to quickly arrive at the same starting point represented by (8). The change of potential energy that occurs from angular position \( \alpha \) to \( \varphi \) can be equated to the increase in kinetic energy (since the bob is momentarily motionless at angular position \( \alpha \)) as

\[
\frac{1}{2}mv^2 = mgR \left[ \cos(\varphi) - \cos(\alpha) \right] \tag{15}
\]

Since the velocity \( v \) must be tangential to the arc that is scribed by the bob, at any instant in time \( v = R \left( \frac{d\varphi}{dt} \right) \). Substituting this into (15) leads directly to (8).

The elliptic integral of the first kind is generally presented as

\[
F(k,x) = \int_{0}^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2(\theta)}} \tag{16}
\]

with the complete elliptic integral of the first kind given by \( F(k,\pi/2) \). It is easy to show that

\[
T = 2\pi \sqrt{\frac{R}{g}} \left[ 1 + \left( \frac{1}{2} \right)^2 k^2 + \left( \frac{1 \cdot 3}{2 \cdot 4} \right)^2 k^4 + \left( \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \right)^2 k^6 + \ldots \right] \tag{17}
\]
Straight forward visual inspection of (17) easily shows that the series is slow to converge when $k$ is reasonably close to unity.

3. Accurate Computation of the Elliptic Integral of the First Kind

Gauss’s Transformation\textsuperscript{2} can be used to expand the elliptic integral (16) into an expansion where

$$ F(\phi, k) = (1 + k_1) F(\phi_1, k_1) \quad (18) $$

This expansion can be repeatedly applied ultimately leading in the limit to $\lim_{p \to \infty} F(\phi_p, k_p) = \frac{\pi}{2}$. The expansion generally converges to 10 or more decimal place accuracy within only a few recursions of (18).

The other formulas that accompany (18) are the following:

$$
\begin{align*}
    k' &= \sqrt{1 - k^2} \\
    k_1 &= \frac{1 - k'}{1 + k'} \\
    \phi_1 &= \arcsin \left[ \frac{(1 + k') \sin(\phi)}{1 + \sqrt{1 - k^2 \sin^2(\phi)}} \right]
\end{align*}
$$

(19)

In the case where the complete elliptic integral of the first kind is to be computed (i.e., $\phi = \frac{\pi}{2}$), a different set of recursive formulas \[7\] can be used to compute the desired result with even less effort as given by

\[ a_0 = 1 + k \]
\[ b_0 = 1 - k \]

Recursively Compute:

\[ a_{i+1} = \frac{a_i + b_i}{2} \]
\[ b_{i+1} = \sqrt{a_i b_i} \]

Upon Convergence:

\[ F\left(\frac{\pi}{2}, k\right) = \frac{\pi}{2a_n} \]

4. Comparison or Linearized Model Results with Ideal

All of the mathematics are greatly simplified if the linearized model represented by (4) is used rather than the complete nonlinear model. For the linearized case, the frequency of the pendulum’s motion is exactly computable as (5) and the pendulum’s motion is precisely sinusoidal.

For a very large range of starting phases, the pendulum’s motion is very closely approximated by a sinusoid assuming the time period given by (14). In all but the most rigorous cases, this is in all likelihood adequately precise.

The appreciation for the linear differential equation represented by (4) is quickly appreciated over the nonlinear differential equation (3) when implicit and or higher-order numerical solutions of the differential equation are desired for greater accuracy. The author has frequently used the second-order Gear method \[5\] with good success, but this formulation is not possible with the nonlinear differential equation (3).

5. Numerical Solution of the Differential Equations

The differential equation (3) solution may be computed numerically in the time domain.

5.1 Forward Euler Integration

Although prone to accuracy and stability issues, the forward Euler method is often used for

\textsuperscript{2} Also referred to as Landen’s Transformation
solving differential equations because it is extremely simple to use. The forward Euler method is an explicit integration method \[5-6\]. In this case, the time-derivative is approximated as

\[
s'(t) \approx \frac{s(t+h) - s(t)}{h} \tag{21}
\]

where the time increment is given by \(h\). Focusing on the starting differential equation (3), it is simple to re-cast this second-order differential equation as a pair of first-order differential equations by defining

\[
U_1(t) = \phi(t)
\]

\[
U_2(t) = \frac{d\phi}{dt}
\tag{22}
\]

leading to

\[
\frac{dU_2}{dt} = -\frac{g}{R} \sin(U_1)
\]

\[
\frac{dU_1}{dt} = U_2
\tag{23}
\]

Substituting (21) into (23) results in

\[
\frac{U_{2,n+1} - U_{2,n}}{h} = -\frac{g}{R} \sin(U_{1,n})
\]

\[
\frac{U_{1,n+1} - U_{1,n}}{h} = U_{1,n}
\tag{24}
\]

where the index \(n\) represents the value of the parameter at time \(t= nh\) where \(h\) is the constant time step used. Solving (24) for the parameter values at the next time step \(n+1\) produces

\[
U_{2,n+1} = U_{2,n} + \left( -\frac{gh}{R} \right) \sin(U_{1,n})
\]

\[
U_{1,n+1} = U_{1,n} + U_{2,n} h
\tag{25}
\]

The fact that the forward Euler method is an explicit method results in only time-index \(n\) values being on the right side of the equal side and the \(n+1\) (future) time-index values being on the left-hand side.

The set of difference equations can be easily programmed and in the case of \(R= 1\) meter and \(\alpha= 30\) degrees, the result is as shown in Figure 2. Due to numerical imprecision even with \(h= 6\) msec, the computed solution slowly grows in amplitude rather than remaining constant-envelope as the ideal solution shows. Error propagation with the forward Euler method is so poor that the amplitude growth is difficult to avoid.

**Figure 2 Forward Euler Differential Equation Solution**

5.2 **Backward Euler Integration**

Backward Euler integration is an implicit integration method and as such, it is not possible to use this method unless the differential equation is linearized as in (4). Although this is a short-cut path that we wish to avoid, this path will be considered in order to show the greater stability properties of the backward Euler method as compared to the forward Euler method.

For the backward Euler method we write

\[
U_{2,n+1} = U_{2,n} - \frac{gh}{R} U_{1,n+1}
\]

\[
U_{1,n+1} = U_{1,n} + h U_{2,n+1}
\tag{26}
\]

or in matrix form

\[
\begin{bmatrix}
\frac{gh}{R} & 1 \\
1 & -h
\end{bmatrix}
\begin{bmatrix}
U_{1,n+1} \\
U_{2,n+1}
\end{bmatrix} =
\begin{bmatrix}
U_{2,n} \\
U_{1,n}
\end{bmatrix}
\tag{27}
\]

Solving this for the next-step state-variable values,
This set of simultaneous difference equations can be programmed very easily also leading to the results shown in Figure 3. In the backward Euler case, the

\[
\begin{bmatrix}
U_{1,n+1} \\
U_{2,n+1}
\end{bmatrix} = \begin{bmatrix}
1 & \frac{g\Delta t}{R} \\
1 - \frac{g\Delta t^2}{R}
\end{bmatrix} \begin{bmatrix}
U_{2,n} \\
U_{1,n}
\end{bmatrix}
\] (28)

This set of simultaneous difference equations can be programmed very easily also leading to the results shown in Figure 3. In the backward Euler case, the

Figure 3 Backward Euler Differential Equation Solution

numerical imprecision leads to a decay in the envelope magnitude, so although this is clearly a more stable situation, the extent of the numerical error is about the same as for the forward Euler method.

In the section that follows, we will see that the 4th order Runge-Kutta method is dramatically more accurate and well behaved than either Euler method considered thus far.

5.3 Runge-Kutta Method

The derivation of the Runge-Kutta method is beyond the scope of this memorandum, but interested readers may refer to [4,6]. Results for the second-order and fourth-order Runge-Kutta methods applied to the second-order differential equation (3) follow.

5.3.1 Second-Order Runge-Kutta

The formula for the second-order Runge-Kutta solution to the second-order differential equation are given by

\[
k_1 = f(t_n, x_n, y_n)
\]
\[
j_1 = g(t_n, x_n, y_n)
\]
\[
k_2 = f\left(t_n + \frac{\Delta t}{2}, x_n + \frac{\Delta t}{2} k_1, y_n + \frac{\Delta t}{2} j_1\right)
\]
\[
j_2 = g\left(t_n + \frac{\Delta t}{2}, x_n + \frac{\Delta t}{2} k_1, y_n + \frac{\Delta t}{2} j_1\right)
\]
\[
x_{n+1} = x_n + h k_2
\]
\[
y_{n+1} = y_n + h j_2
\] (29)

In the context of the present set of differential equations,

\[
U_1(t) = \phi(t)
\]
\[
U_2(t) = \frac{d\phi}{dt}
\]
\[
\frac{dU_2}{dt} = f(...) = -\frac{g}{R} \sin(U_1) \tag{30}
\]
\[
\frac{dU_1}{dt} = g(...) = U_2
\]

which leads further to

\[
k_1 = -\frac{g}{R} \sin(U_{1, n})
\]
\[
j_1 = U_{2, n}
\]
\[
k_2 = -\frac{g}{R} \sin\left(U_{1, n} + \frac{\Delta t}{2} j_1\right)
\]
\[
j_2 = U_{2, n} + \frac{\Delta t}{2} k_1
\]
\[
U_{1, n+1} = U_{1, n} + h j_2
\]
\[
U_{2, n+1} = U_{2, n} + h k_2
\] (31)

These finite difference equations are easily programmed and the results for several different time steps are shown in Figure 4 through Figure 6. As shown in these figures, the results follow the exact solution very closely until the time step is increased too far to 200 msec as shown.
in Figure 6 where the onset of some instability is apparent.

**Figure 4 2nd Order Runge-Kutta with h= 30 msec**

![Graph showing 2nd Order Runge-Kutta and Ideal compared](image)

**Figure 5 2nd Order Runge-Kutta with h= 75 msec**

![Graph showing 2nd Order Runge-Kutta and Ideal compared](image)

**Figure 6 2nd Order Runge-Kutta with h= 200 msec**

![Graph showing 2nd Order Runge-Kutta and Ideal compared](image)
5.3.2 Fourth-Order Runge-Kutta

In the case of the 4th-order Runge-Kutta method, the applicable formulas are as follows:

\[
\begin{align*}
    k_1 &= f(t_n, x_n, y_n) \\
    j_1 &= g(t_n, x_n, y_n) \\
    k_2 &= f\left(t_n + \frac{h}{2}, x_n + \frac{h}{2} k_1, y_n + \frac{h}{2} j_1\right) \\
    j_2 &= g\left(t_n + \frac{h}{2}, x_n + \frac{h}{2} k_1, y_n + \frac{h}{2} j_1\right) \\
    k_3 &= f\left(t_n + \frac{h}{2}, x_n + \frac{h}{2} k_2, y_n + \frac{h}{2} j_2\right) \\
    j_3 &= g\left(t_n + \frac{h}{2}, x_n + \frac{h}{2} k_2, y_n + \frac{h}{2} j_2\right) \\
    k_4 &= f\left(t_n + h, x_n + h k_3, y_n + h j_3\right) \\
    j_4 &= g\left(t_n + h, x_n + h k_3, y_n + h j_3\right) \\
    x_{n+1} &= x_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4) \\
    y_{n+1} &= y_n + \frac{h}{6}(j_1 + 2j_2 + 2j_3 + j_4) \quad (32)
\end{align*}
\]

This set of difference equations is easily programmed and the results are shown for several time steps in Figure 7 through Figure 9. As shown in these figures, the computed results match the ideal results almost exactly even at the large time step of 200 msec.

Although other techniques may be superior to the Runge-Kutta methods explored here, the simplicity of the method combined with the very good precision make it a highly recommended method for use in solving differential equations numerically.
6. Connections with Elliptic Filters

Two of the best treatments of elliptic filter design are provided by [2,3,8]. Having been a long admirer of Sidney Darlington’s work with elliptic filters, a number of his related publications are listed here as references [9-13].

A very insightful and unifying view of Butterworth, Chebyshev, and elliptic filters is provided in [9]. Quoting from [9]:

“Formulas for the critical frequencies involved with the design of Butterworth, Chebyshev, and elliptic filters are identical when expressed in terms of appropriate variables. For Butterworth filters, the appropriate variable is simply the frequency \( s = j\omega \). For Chebyshev filters, it is a new variable defined by a simple transformation on \( \omega \). For elliptic filters, the appropriate variable is determined by a sequence of transformations applied recursively, each similar to that for the Chebyshev filters. Interpretation in terms of elliptic function transformations is a possible but unnecessary complication. “

This reference provides the most concise and simple method for calculating the elliptic filter critical frequencies that I am aware of.

7. References

**Advanced Phase-Lock Techniques**

James A. Crawford

2008

Artech House

510 pages, 480 figures, 1200 equations

CD-ROM with all MATLAB scripts


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