# Електродинамика

Андон Рангелов, кабинет В 39, email: rangelov@phys.uni-sofia.bg , интернет страница на курса `http://ed.quantum-bg.org/`

Преобразованията на Лоренц дават връзката между координатите и времето $(x, y, z, t)$ в една инерциална система $S$, спрямо координатите и времето $(x', y', z', t')$ в друга инерциална отправна система $S'$, която се движи със скорост $u$ насочена по оста $x$ на системата $S$.

$$\begin{aligned}
y' &= y, \quad z' = z, \\
x' &= \frac{x - ut}{\sqrt{1 - \frac{u^2}{c^2}}} = \gamma\left(x - ut\right), \\
t' &= \frac{t - \frac{u}{c^2}x}{\sqrt{1 - \frac{u^2}{c^2}}} = \gamma\left(t - \frac{u}{c^2}x\right). \\
\gamma &= \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}
\end{aligned}$$

Като използвате преобразованията на Лоренц покажете че оператора на Даламбер е инвариант (еднакъв за всички инерциални отправни системи).
Оператора на Даламбер $\Box$ се дефинира по следният начин.

$$\Box = \triangle - \frac{1}{c^2}\frac{\partial^2}{\partial t^2},$$

Където $\triangle$ е оператор на Лаплас.

**Решение**

Записваме оператора на Даламбер в Декартова координатна система

$$\Box = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$$

от преобразованията на Лоренц лесно може да съобразим, че

$$\begin{aligned}
\frac{\partial}{\partial y} &= \frac{\partial}{\partial y'} \\
\frac{\partial^2}{\partial y^2} &= \frac{\partial^2}{\partial y'^2} \\
\frac{\partial}{\partial z} &= \frac{\partial}{\partial z} \\
\frac{\partial^2}{\partial z^2} &= \frac{\partial^2}{\partial z'^2}
\end{aligned}$$

докато $x(x', t')$ и $t(x', t')$ тоест ползвайки диференциране на функция от функция имаме

$$\begin{aligned}
\frac{\partial}{\partial x} &= \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} + \frac{\partial t'}{\partial x}\frac{\partial}{\partial t'} \\
\frac{\partial}{\partial t} &= \frac{\partial x'}{\partial t}\frac{\partial}{\partial x'} + \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'}
\end{aligned}$$

и тогава за вторите производни имаме

$$\begin{aligned}
\frac{\partial^2}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} + \frac{\partial t'}{\partial x}\frac{\partial}{\partial t'}\right) = \\
&= \frac{\partial^2 x'}{\partial x^2}\frac{\partial}{\partial x'} + \frac{\partial x'}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x'}\right) + \frac{\partial^2 t'}{\partial x^2}\frac{\partial}{\partial t'} + \frac{\partial t'}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t'}\right)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2}{\partial t^2} &= \frac{\partial}{\partial t}\left(\frac{\partial}{\partial t}\right) = \frac{\partial}{\partial t}\left(\frac{\partial x'}{\partial t}\frac{\partial}{\partial x'} + \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'}\right) = \\
&= \frac{\partial^2 x'}{\partial t^2}\frac{\partial}{\partial x'} + \frac{\partial x'}{\partial t}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial x'}\right) + \frac{\partial^2 t'}{\partial t^2}\frac{\partial}{\partial t'} + \frac{\partial t'}{\partial t}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t'}\right)
\end{aligned}$$

сега отчитаме, че преобразованията на Лоренц са линейни и тогава $\frac{\partial^2 x'}{\partial x^2} = \frac{\partial^2 t'}{\partial t^2} = \frac{\partial^2 x'}{\partial t^2} = \frac{\partial^2 t'}{\partial x^2} = 0 \Rightarrow$

$$\frac{\partial^2}{\partial x^2} = \frac{\partial x'}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x'}\right) + \frac{\partial t'}{\partial x}\frac{\partial}{\partial x}\left(\frac{\partial}{\partial t'}\right)$$

$$\frac{\partial^2}{\partial t^2} = \frac{\partial x'}{\partial t}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial x'}\right) + \frac{\partial t'}{\partial t}\frac{\partial}{\partial t}\left(\frac{\partial}{\partial t'}\right)$$

замествайки в горните две уравнения производните $\frac{\partial}{\partial x} = \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} + \frac{\partial t'}{\partial x}\frac{\partial}{\partial t'}$ и $\frac{\partial}{\partial t} = \frac{\partial x'}{\partial t}\frac{\partial}{\partial x'} + \frac{\partial t'}{\partial t}\frac{\partial}{\partial t'} \Rightarrow$

$$\frac{\partial^2}{\partial x^2} = \frac{\partial x'}{\partial x}\left(\frac{\partial x'}{\partial x}\frac{\partial^2}{\partial x'^2} + \frac{\partial t'}{\partial x}\frac{\partial^2}{\partial t'\partial x'}\right) + \frac{\partial t'}{\partial x}\left(\frac{\partial x'}{\partial x}\frac{\partial^2}{\partial x'\partial t'} + \frac{\partial t'}{\partial x}\frac{\partial^2}{\partial t'^2}\right)$$

$$\frac{\partial^2}{\partial t^2} = \frac{\partial x'}{\partial t}\left(\frac{\partial x'}{\partial t}\frac{\partial^2}{\partial x'^2} + \frac{\partial t'}{\partial t}\frac{\partial^2}{\partial t'\partial x'}\right) + \frac{\partial t'}{\partial t}\left(\frac{\partial x'}{\partial t}\frac{\partial^2}{\partial x'\partial t'} + \frac{\partial t'}{\partial t}\frac{\partial^2}{\partial t'^2}\right)$$

от преобразованията на Лоренц имаме

$$\begin{aligned}\frac{\partial x'}{\partial t} &= -u\gamma \\ \frac{\partial t'}{\partial t} &= \gamma \\ \frac{\partial x'}{\partial x} &= \gamma \\ \frac{\partial t'}{\partial x} &= -\frac{u}{c^2}\gamma\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}\frac{\partial^2}{\partial t^2} &= -u\gamma\left(-u\gamma\frac{\partial^2}{\partial x'^2} + \gamma\frac{\partial^2}{\partial t'\partial x'}\right) + \gamma\left(-u\gamma\frac{\partial^2}{\partial x'\partial t'} + \gamma\frac{\partial^2}{\partial t'^2}\right) = \\ &= u^2\gamma^2\frac{\partial^2}{\partial x'^2} - 2u\gamma^2\frac{\partial^2}{\partial t'\partial x'} + \gamma^2\frac{\partial^2}{\partial t'^2}\end{aligned}$$

$$\begin{aligned}\frac{\partial^2}{\partial x^2} &= \gamma\left(\gamma\frac{\partial^2}{\partial x'^2} - \frac{u}{c^2}\gamma\frac{\partial^2}{\partial t'\partial x'}\right) - \frac{u}{c^2}\gamma\left(\gamma\frac{\partial^2}{\partial x'\partial t'} - \frac{u}{c^2}\gamma\frac{\partial^2}{\partial t'^2}\right) = \\ &= \gamma^2\frac{\partial^2}{\partial x'^2} + \frac{u^2}{c^4}\gamma^2\frac{\partial^2}{\partial t'^2} - 2\frac{u}{c^2}\gamma^2\frac{\partial^2}{\partial t'\partial x'}\end{aligned}$$

$$\begin{aligned}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} &= \frac{u\gamma}{c^2}\left(-u\gamma\frac{\partial^2}{\partial x'^2} + \gamma\frac{\partial^2}{\partial t'\partial x'}\right) - \frac{\gamma}{c^2}\left(-u\gamma\frac{\partial^2}{\partial x'\partial t'} + \gamma\frac{\partial^2}{\partial t'^2}\right) = \\ &= 2\frac{u\gamma^2}{c^2}\frac{\partial^2}{\partial t'\partial x'} - \frac{u^2\gamma^2}{c^2}\frac{\partial^2}{\partial x'^2} - \frac{\gamma^2}{c^2}\frac{\partial^2}{\partial t'^2}\end{aligned}$$

тогава оператора на Даламбер в Декартова координатна система става

$$\begin{aligned}\Box &= \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} = \\ &= \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2} + \gamma^2\frac{\partial^2}{\partial x'^2} + \frac{u^2}{c^2}\gamma^2\frac{\partial^2}{\partial t'^2} - \frac{u^2\gamma^2}{c^2}\frac{\partial^2}{\partial x'^2} - \frac{\gamma^2}{c^2}\frac{\partial^2}{\partial t'^2} = \\ &= \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2} + \gamma^2\frac{\partial^2}{\partial x'^2}\left(1 - \frac{u^2}{c^2}\right) - \gamma^2\frac{1}{c^2}\frac{\partial^2}{\partial t'^2}\left(1 - \frac{u^2}{c^2}\right) = \\ &= \frac{\partial^2}{\partial y'^2} + \frac{\partial^2}{\partial z'^2} + \frac{\partial^2}{\partial x'^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2} = \Box'\end{aligned}$$